Selected Solutions (§15.9)

(26) Compute $\iint_R (4x + 8y)\,dA$ for R the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, $(1,5)$ via transformation $(x,y) = \left(\frac{1}{4}(u+v), \frac{1}{4}(v-3u)\right)$.

Sol: Given the transformation, we need to compute the new region. We do that by computing the inverse transformation and applying it to points:

$$\begin{cases} x = \frac{1}{4}(u+v) \\ y = \frac{1}{4}(v-3u) \end{cases} \iff \begin{cases} u+v = 4x & (1) \\ -3u+v = 4y & (2) \end{cases} \iff \begin{cases} 4u = 4x - 4y & \leftarrow (1)-(2) \\ 4v = 12x + 4y & \leftarrow 3\times(1)+(2) \end{cases}$$

$$\iff \begin{cases} u = x - y \\ v = 3x + y \end{cases}$$

We compute the region via $(x,y) \longmapsto (u,v) = (x-y, 3x+y)$

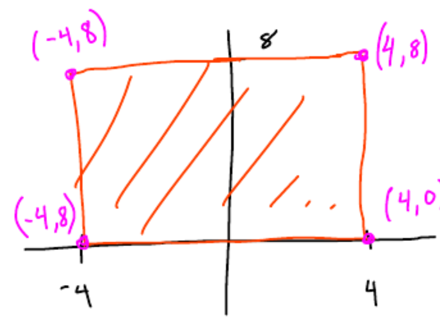
$(-1,3) \longmapsto (-1-3, -3+3) = (-4,0)$

$(1,-3) \longmapsto (1+3, 3-3) = (4,0)$

$(3,-1) \longmapsto (3+1, 9-1) = (4,8)$

$(1,5) \longmapsto (1-5, 3+5) = (-4,8)$

Hence the desired region is the square to the right



Hence $R_{u,v} = \{(u,v) : -4 \le u \le 4,\ 0 \le v \le 8\} = [-4,4] \times [0,8]$

The Jacobian of the given transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = \det\begin{bmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{bmatrix} = \frac{1}{4}\cdot\frac{1}{4} - \frac{-3}{4}\cdot\frac{1}{4} = \frac{4}{16} = \frac{1}{4}.$$

Finally we compute the integral:

$$\iint_R (4x+8y)\,dA_{x,y} = \iint_R \left(4\cdot\frac{1}{4}(u+v) + 8\cdot\frac{1}{4}(v-3u)\right)\cdot\frac{1}{4}\,dA_{u,v}$$

$$= \frac{1}{4}\int_{u=-4}^{4}\int_{v=0}^{8}(-5u+3v)\,dv\,du = \frac{1}{4}\int_{u=-4}^{4}\left[\frac{3}{2}v^2 - 5uv\right]_{v=0}^{8}du$$

$$= \frac{1}{4}\int_{u=-4}^{4}\left(\frac{3}{2}\cdot 64 - 40u\right)du = \int_{u=-4}^{4}(24-10u)\,du = \left[24u - 5u^2\right]_{u=-4}^{4}$$

$$= (24\cdot 4 - 5\cdot 16) - (24\cdot -4 - 5\cdot 16) = 24\cdot 8 = 192$$

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